Euler's method applied to the control of switched systems

Laurent Fribourg[1]

joint work with: Adrien Le Coënt[2], Florian De Vuyst[2], Ludovic Chamoin[3], Julien Alexandre dit Sandretto[4], Alexandre Chapoutot[4]

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Plan

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   $\longrightarrow$ basic problem of one-step invariance and set (or symbolic) integration

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   $\longrightarrow$ basic problem of one-step invariance and set (or symbolic) integration
3. Euler's method and error estimation δ (using Dahlquist constant λ)
   $\longrightarrow$ application to set integration and control synthesis
4. Euler error in presence of disturbance
   $\longrightarrow$ application to distributed control synthesis
5. Comparison with classical interval-based integration methods

Outline

Switched systems

A continuous switched system

$$\dot{x}(t) = f_{\sigma(t)}(x(t))$$

- state $x(t) \in \mathbb{R}^n$
- switching rule $\sigma(\cdot) : \mathbb{R}^+ \longrightarrow U$
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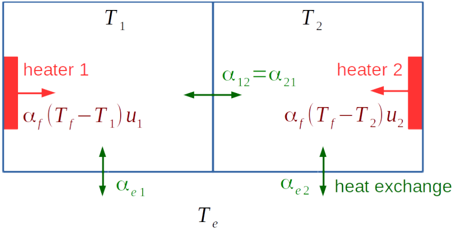
Control Synthesis problem:
Find at each sampling time, the appropriate mode $u \in U$ (in function of the value of $x(t)$) in order to make the system satisfy a certain property.

Example: Two-room apartment



$$\begin{pmatrix}\dot{T}_1\\ \dot{T}_2\end{pmatrix} = \begin{pmatrix}-\alpha_{21}-\alpha_{e1}-\alpha_f u_1 & \alpha_{21}\\ \alpha_{12} & -\alpha_{12}-\alpha_{e2}-\alpha_f u_2\end{pmatrix}\begin{pmatrix}T_1\\ T_2\end{pmatrix}+\begin{pmatrix}\alpha_{e1}T_e+\alpha_f T_f u_1\\ \alpha_{e2}T_e+\alpha_f T_f u_2\end{pmatrix}.$$

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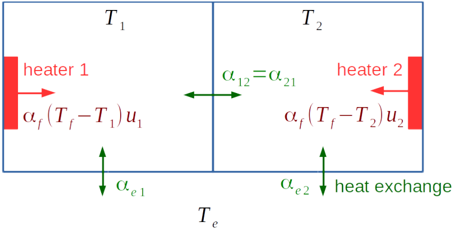

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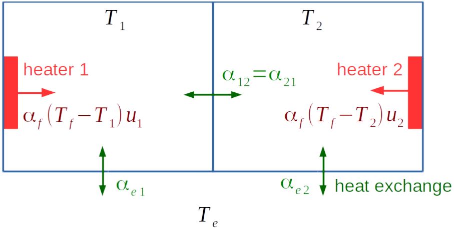

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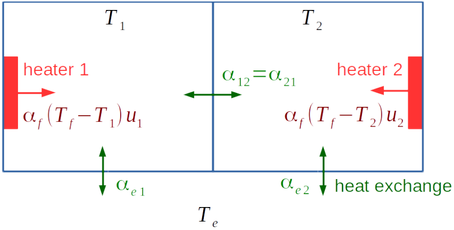

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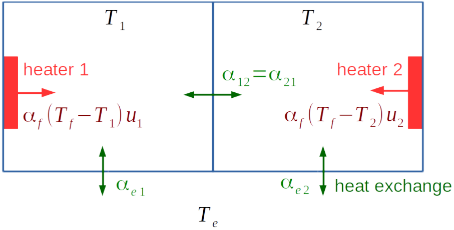

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- A state dependent control consists in selecting at each τ a mode (or a pattern) according to the current value of the state.

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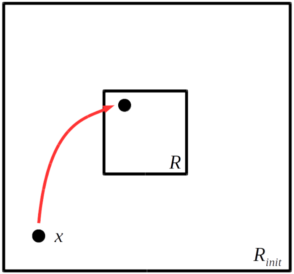

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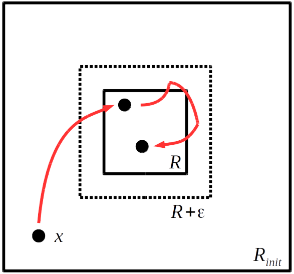

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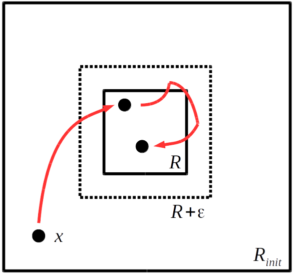

NB: classic stabilization to an equilibrium point, impossible to achieve here
$\leadsto$ practical stability

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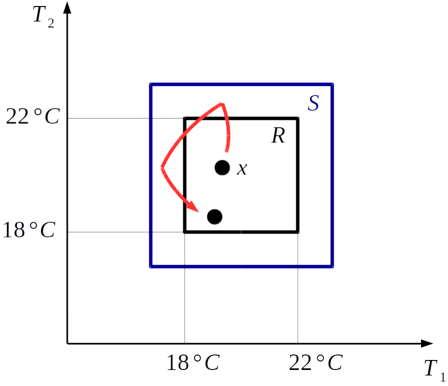
Focus on (R, S)-stability



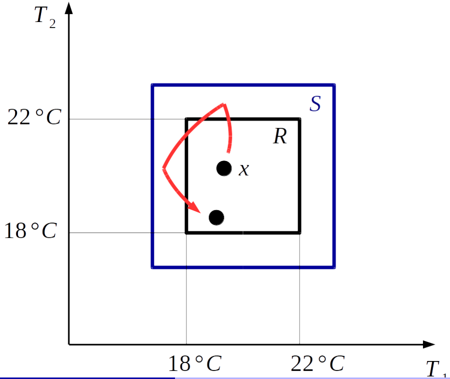
Focus on (R, S)-stability

Being given a recurrence (rectang.) set R and a safety (rectang.) set S, we consider the state-dependent control problem of synthesizing σ:

At each sampling time t, determine the switched mode $u \in U$ in function of the value of $x(t)$, in order to satisfy:

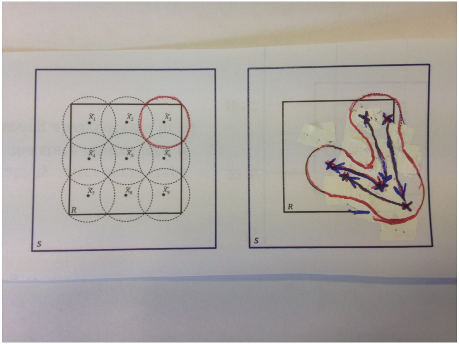
(R,S)-stability:
if $x(0) \in R$, then $x(t)$:

1. returns infinitely often into R, and
2. always stays in S.



Principle of (R,S)-stability control synthesis (MINIMATOR)

[R. Soulat's PhD, 2013]



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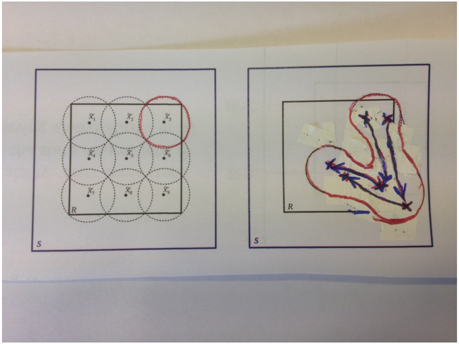

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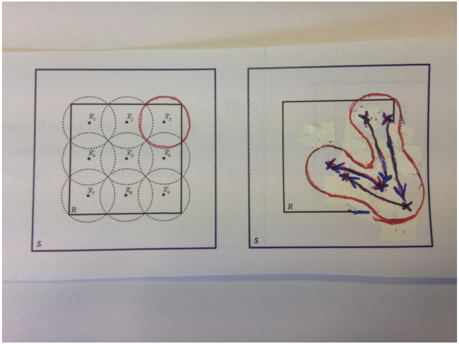

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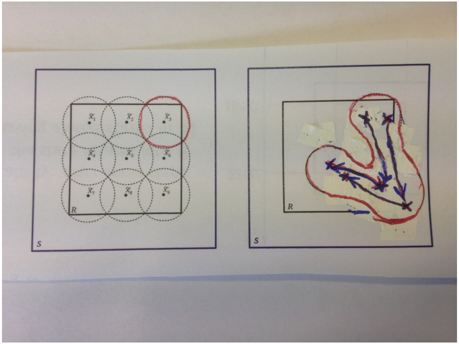

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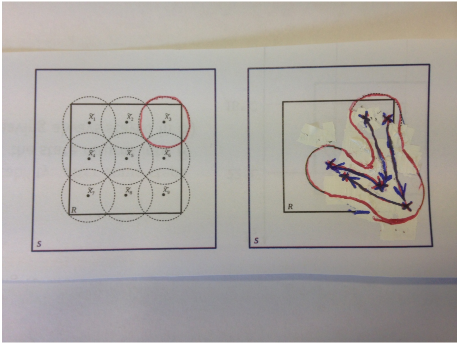

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Remarks on (R, S)-stability control

1. At each ball B (covering R), is assoc. a "returning pattern" π of lg., say k

[1] $\approx$ Model Predictive Control where the optimal strategy is estimated (online) for the next k_1 steps (but strategy updated there at *each* step, $\neq$ after k_1 steps: "receding prediction horizon"). Note also that, here, control π_1 is computed off line.

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$\rightarrow$ exponential in n, d, K
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4. The length $|\pi_1| = k_1$ can be seen as a time-horizon: the strategy is planned for k_1 steps, then updated after k_1 steps.[1]

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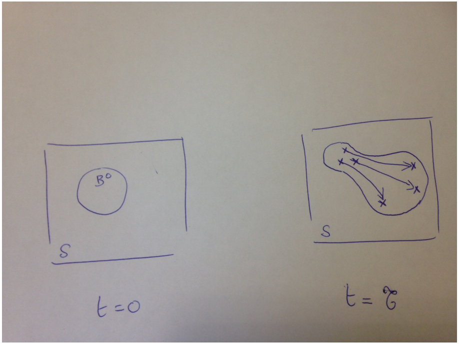
Key notion: one-step invariance

Given a ball $B^0 \subset S$, find a mode $u \in U$ s.t. all the u-trajectories $x(t)$ with $x(0) \in B^0$, satisfy:

$$x(t) \in S \quad \text{for all } t \in [0, \tau]$$



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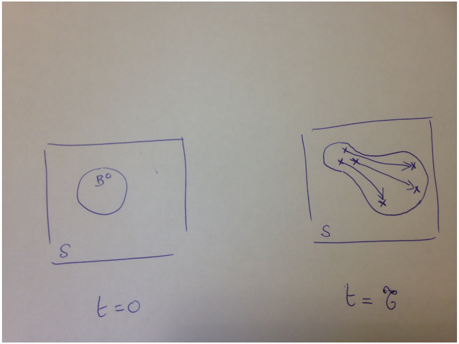
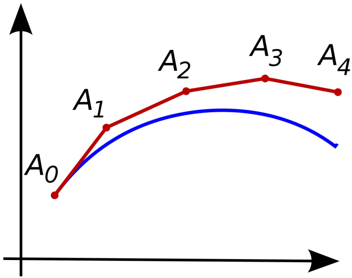


⟶ requires a techn. of set-integration; we will use an Euler-based techn.

Outline

Euler's estimation method of $x(t)$ (with $\dot{x}(t) = f(x(t))$)

$$\tilde{x}(t) = \tilde{x}(t_0) + f(\tilde{x}(t_0))(t - t_0)$$



Suppose that, for the current step size τ (or a sub-sampling size h), the derivative is constant and equal to the derivative at the starting point

Global error estimated with Lipschitz constant L

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We now consider a more appropriate constant λ that leads to sharper estimations of the Euler error.

Dahlquist's constant λ ("one-sided Lipschitz" constant)

- $\lambda \in \mathbb{R}$ is a constant s.t., for all $x, y \in S$:

$$\langle f(y) - f(x), y - x \rangle \leq \lambda \|y - x\|^2$$

where $\langle \cdot, \cdot \rangle$ denote the scalar product of two vectors of $\mathbb{R}^n$

[2] Define $V(x, x') = \|x - x'\|^2$; we have: $\frac{dV}{dt} \leq \lambda V$ (hence $V = V_0 e^{\lambda t}$). So V is an exponentially stable Lyapunov function when $\lambda < 0$.

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Local error function $\delta(\cdot)$ estimated using constant λ

Given an initial error δ_0 of $\tilde{x}(t)$ (i.e.: $\|\tilde{x}(0) - x(0)\| \leq \delta_0$),
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- if $\lambda < 0$:

$$\delta(t) = \left(\delta_0^2 e^{\lambda t} + \frac{C^2}{\lambda^2} \left(t^2 + \frac{2t}{\lambda} + \frac{2}{\lambda^2} \left(1 - e^{\lambda t} \right) \right) \right)^{\frac{1}{2}}$$

- if $\lambda = 0$:

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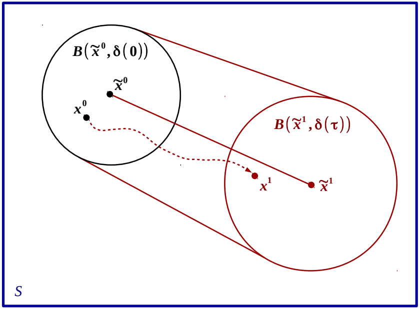
see [A. Le Coënt's Ph.D Thesis, 2017].

One-step invariance using the E. error fn $\delta(\cdot)$

- Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subset S$, find a mode u s.t.:

$$x(t) \in S \quad \text{for all } x(0) \in B^0, t \in [0, \tau]$$

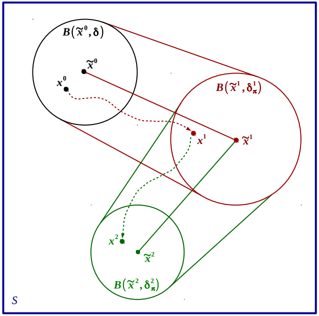


i.e.: $B^1 \equiv B(\tilde{x}^1, \delta^1) \subset S$ with $\tilde{x}^1 = \tilde{x}^0 + f(\tilde{x}^0)\tau$ and $\delta^1 = \delta(\tau)$ (assuming convexity of $\delta(\cdot)$ on $[0, \tau]$).

Finding a control pattern π using $\delta(\cdot)$

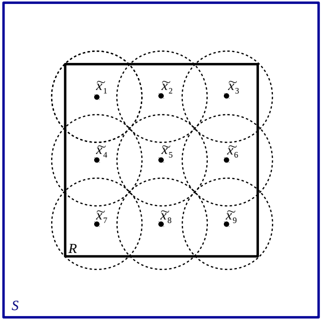
- Given a ball $B^0 \equiv B(\tilde{x}^0, \delta^0) \subset S$, find a pattern π (of length k) s.t.:

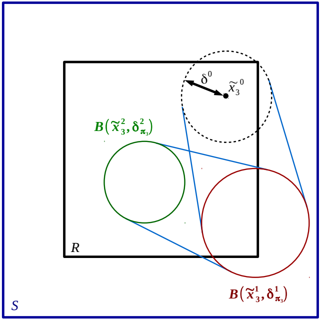
$$x(t) \in S \quad \text{for all } x(0) \in B^0, t \in [0, k\tau]$$



i.e.: $B^1 \equiv B(\tilde{x}^1, \delta^1) \subset S, \quad \ldots, \quad B^k \equiv B(\tilde{x}^k, \delta^k) \subset S$

(R,S)-stable control synthesis using E. error fn $\delta(\cdot)$





For each ball $B_i^0 \equiv B(\tilde{x}_i^0, \delta_i^0) \subset S$ covering R, find a pattern π_i (of length k_i) s.t.:

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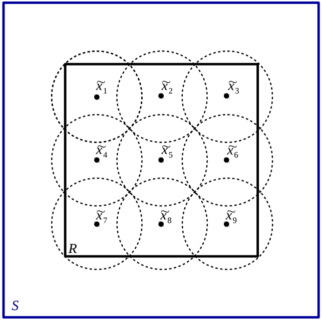

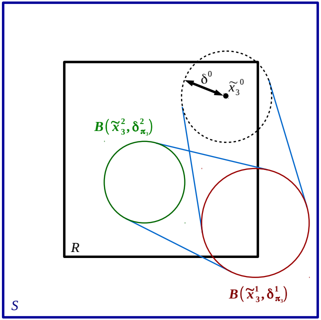

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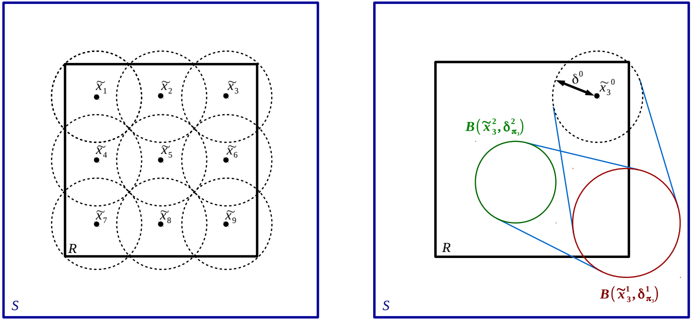
- Safety: $B_i^1 \equiv B(\tilde{x}_i^1, \delta_i^1) \subset S, ..., B_i^{k_i-1} \equiv B(\tilde{x}_i^{k_i-1}, \delta_i^{k_i-1}) \subset S$, and

(R,S)-stable control synthesis using E. error fn $\delta(\cdot)$



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- Recurrence: $B_i^{k_i} \equiv B(\tilde{x}_i^{k_i}, \delta_i^{k_i}) \subset R$

Outline

incremental Input-to-State Stability (i-ISS) in presence of disturbance $w \in W$

Consider: $\dot{x}(t) = f(x(t), w(t))$ with $w(t) \in W$ for all $t \in [0, \tau]$.

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[3] In case $\lambda < 0$, (H) expresses (a variant of) the fact that $V(x, x') = \|x - x'\|^2$ is an i-ISS Lyapunov fn (see, e.g., [D. Angeli] [Hespanha et al.]). The constants λ, γ can be numerically computed using constrained optimization algos.

incremental Input-to-State Stability (i-ISS) in presence of disturbance $w \in W$

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The eq. $\dot{x} = f(x, w)$ with $w \in W$ is said to satisfy the property of i-ISS w.r.t disturbance set W if
$\exists\ \lambda \in \mathbb{R}$ [3] and $\gamma \in \mathbb{R}_{\geq 0}$ s.t.

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(H) $\forall x, x' \in S$, $\forall w, w' \in W$:

$$\langle f(x, w) - f(x', w'), x - x' \rangle \leq \lambda \|x - x'\|^2 + \gamma \|x - x'\| \|w - w'\|.$$

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E. error function $\delta_W(\cdot)$ in presence of disturbance $w \in W$

- Consider the ODE:
  $\dot{x}(t) = f(x(t), w(t)) \quad \text{with} \quad w(t) \in W \quad \text{for all } t \in [0, \tau].$

E. error function $\delta_W(\cdot)$ in presence of disturbance $w \in W$

- The fn $\delta_W(\cdot)$ (s.t: for all $t \in [0,\tau]$, $w(t) \in W$: $\|x(t) - \tilde{x}(t)\| \leq \delta_W(t)$) can now be defined by:
  - if $\lambda < 0$,

$$\delta_W(t) = \left(\frac{(C)^2}{-(\lambda)^4} \left(-(\lambda)^2 t^2 - 2\lambda t + 2e^{\lambda t} - 2 \right) + \frac{1}{(\lambda)^2} \left(\frac{C\gamma|W|}{-\lambda} \left(-\lambda t + e^{\lambda t} - 1 \right) + \lambda \left(\frac{(\gamma)^2 (|W|/2)^2}{-\lambda} (e^{\lambda t} - 1) + \lambda (\delta^0)^2 e^{\lambda t} \right) \right) \right)^{1/2} \quad (1)$$

  - if $\lambda > 0$,

$$\delta_W(t) = \frac{1}{(3\lambda)^{3/2}} \left(\frac{C^2}{\lambda} \left(-9(\lambda)^2 t^2 - 6\lambda t + 2e^{3\lambda t} - 2 \right) + 3\lambda \left(\frac{C\gamma|W|}{\lambda} \left(-3\lambda t + e^{3\lambda t} - 1 \right) + 3\lambda \left(\frac{(\gamma)^2 (|W|/2)^2}{\lambda} (e^{3\lambda t} - 1) + 3\lambda (\delta^0)^2 e^{3\lambda t} \right) \right) \right)^{1/2} \quad (2)$$

  - if $\lambda = 0$,

$$\delta_W(t) = \left((C)^2 \left(-t^2 - 2t + 2e^t - 2 \right) + \left(C\gamma|W| \left(-t + e^t - 1 \right) + ((\gamma)^2 (|W|/2)^2 (e^t - 1) + (\delta^0)^2 e^t) \right) \right)^{1/2}$$

Compositional (R,S)-stability using E. error fns $\delta_{1,S_2}, \delta_{2,S_1}$

$$\dot{x}_1 = f^1(x_1, x_2)$$
$$\dot{x}_2 = f^2(x_1, x_2)$$

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Suppose:

- (H1) $\dot{x}_1 = f^1(x_1, x_2)$ is i-ISS w.r.t disturbance $x_2 \in S_2$, with λ^1, γ^1.
- (H2) $\dot{x}_2 = f^2(x_1, x_2)$ is i-ISS w.r.t disturbance $x_1 \in S_1$, with λ^2, γ^2.

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Theorem (compositionality): If

- σ_1 is an (R_1, S_1)-stable control of $x_1(t)$ with S_2 as domain of disturbance, using the E. error fn $\delta_{1,S_2}(t)$ (bounding $\|\tilde{x}_1(t) - x_1(t)\|$ in terms of (λ^1, γ^1))
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Compositional (R,S)-stability using E. error fns $\delta_{1,S_2}, \delta_{2,S_1}$

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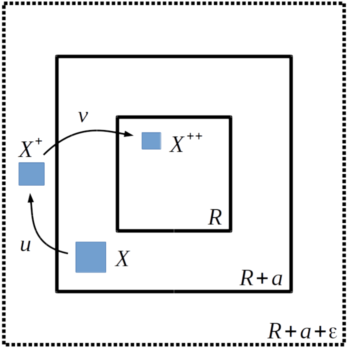
Then: $\sigma = \sigma_1 | \sigma_2$ is an $(R_1 \times R_2, S_1 \times S_2)$-stable control of $x(t) = (x_1(t), x_2(t))$.

Illustration of Distributed vs. Centralized Control

Centralized control synthesis

$$\dot{x}(t) = f_u(x(t))$$

Example of a validated pattern of length 2 mapping the "ball" X into R with $S = R + a + \varepsilon$ as safety box:



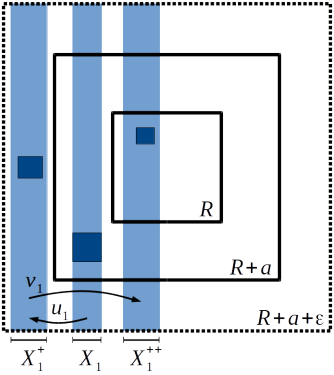
- $X \subset R$
- $X^+ = f_u(X) \subset S$
- $X^{++} = f_v(X^+) \subset R$

- Pattern $u \cdot v$ depends on X

Distrib. Control Synth. (of x_1 using S_2 as approx. of x_2)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
$$\dot{x}_2(t) = f^2_{u_2}(x_1(t), x_2(t))$$

Target zone: $R = R_1 \times R_2$



- $X_1 \subset R_1$
- $X_1^+ = f^1_{u_1}(X_1, S_2) \subset S_1$
- $X_1^{++} = f^1_{v_1}(X_1^+, S_2) \subset R_1$

- Pattern $u_1 \cdot v_1$ depends only on X_1

Distrib. Control Synth. (of x_2 using S_1 as approx. of x_1)

$$\dot{x}_1(t) = f^1_{u_1}(x_1(t), x_2(t))$$
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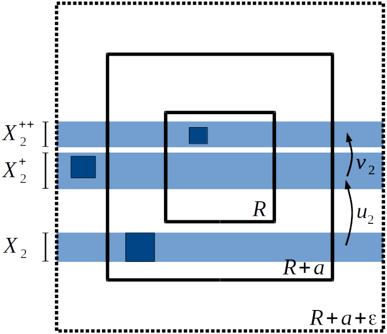
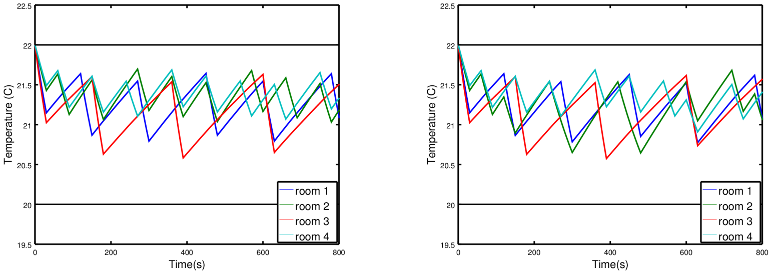
Target zone: $R = R_1 \times R_2$



- $X_2 \subset R_2$
- $X_2^+ = f^2_{u_2}(S_1, X_2) \in S_2$
- $X_2^{++} = f^2_{v_2}(S_1, X_2^+) \in R_2$
- Pattern $u_2 \cdot v_2$ depends only on X_2

Application to distributed control of switched systems



Simulations of centralized control (left) and distributed (right) for the 4-rooms problem [P.-J. Meyer's Ph.D., 2015]

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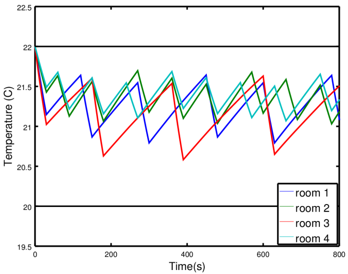

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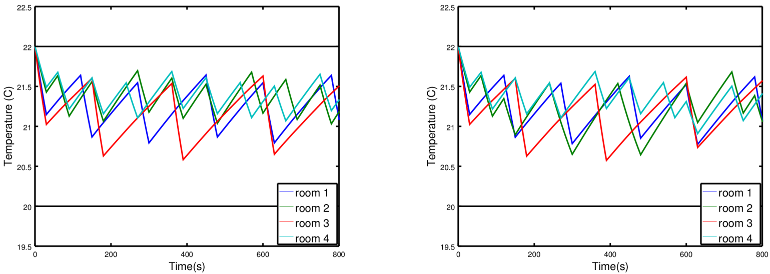

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- centralized synthesis ($|\pi| = 2$): sub-sampling $h = \frac{\tau}{20}$,
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- distributed synthesis ($|\pi| = 2$): sub-sampling $h = \frac{\tau}{10} \mid h = \frac{\tau}{1}$,
  $2^2 \mid 2^2$ modes, $16 \mid 16$ balls $\quad\rightarrow\quad$ $< 1\ s$. of CPU time.

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7. Deserves further experimentations...

Thanks!